EXTREMAL PROBLEMS AND HARMONIC INTERPOLATION ON OPEN RIEMANN SURFACES(1)

BY

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INTRODUCTION

1. The ultimate purpose of the present paper is to study interpolation of harmonic and analytic functions on open Riemann surfaces W. We shall, however, first take a less restricted viewpoint and consider, in general, extremal problems on Riemann surfaces.

Extrema of given functionals can often be immediately found in the special case where W is compact and bounded by a finite number of analytic Jordan curves. The difficulties arise when the contours are arbitrary or, as is always the case for an infinite genus, the boundary exists in an ideal sense only. With this in view, we shall start with a theorem for reducing extremal problems from the general case of an open Riemann surface W to the special case just mentioned. The content of the reduction theorem (no. 5) will be, roughly speaking, as follows:

If the functional to be minimized increases with the region, and if the minimizing functions on compact subregions W_n with analytic boundaries form a normal family, then the solvability of the extremal problem for W_n implies that for W.

2. The reduction theorem will enable us to establish a rather general theorem on minimizing functionals of harmonic functions on Riemann surfaces. We consider the class of single-valued harmonic functions p with a finite number of prescribed poles or logarithmic singularities ζ_i $(i=1, \cdots, n)$ on W. The expansions of p about ζ_i have the form

$$p = \text{Re}\left\{\sum_0^{\infty} a_\nu z^\nu + \lambda\left(\sum_1^{m} b_\nu z^{-\nu} - c \log |z|\right)\right\}, \tag{1}$$

where a_ν, b_ν are complex, c real coefficients, λ a real parameter, and z the local uniformizer. The following combination of the coefficients will be instrumental:

$$\mu = 2\pi \sum_{i=1}^{n} \text{Re}\left[ca_0 + \sum_{\nu=1}^{m} \nu b_\nu a_\nu\right]. \tag{2}$$

For fixed b_ν, c and subsequently fixed λ, the class $\{p\}$ shall be designated

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by C_λ. We set $\lambda = h+k$, with real h, k, and choose

$$m(p) = \int_\beta p\,dp^* + (k - h)\mu \tag{3}$$

for the functional to be minimized. Here the asterisk denotes the harmonic conjugate, β stands for the ideal boundary, and the integral is understood as the limit of integrals along the boundaries of exhausting subregions. The theorem to be established (no. 7) follows:

There exists a unique function p_{hk} in C_λ which minimizes $m(p)$. For varying h, k, the p_{hk} are interrelated by

$$p_{hk} = hp_0 + kp_1, \tag{4}$$

where p_0, p_1 stand for $p_{1,0}$, $p_{0,1}$, respectively. The minimum value of $m(p)$ is $k^2\mu_1 - h^2\mu_0$, with $\mu_0 = \mu(p_0)$, $\mu_1 = \mu(p_1)$, and the deviation from this minimum is given by the Dirichlet integral of $p - p_{hk}$:

$$m(p) = k^2\mu_1 - h^2\mu_0 + D(p - p_{hk}). \tag{5}$$

It is of interest that here p_0, p_1 are the *principal functions* [16; 18] which correspond to the normal linear operators L_0, L_1, characterized by vanishing normal derivatives and constant boundary values, respectively. By fixing b_ν, c, λ, we can choose the class C_λ of functions (1); by subsequently fixing h, k within $h+k=\lambda$, we can select the functional (3) to be minimized in C_λ. Thus the linear combinations $hp_0 + kp_1$ of the principal functions furnish the solutions to extremal problems of great variety and generality. A detailed account of these applications will be given in [19]. In the present paper we shall make use of the theorem in interpolation problems.

3. Let $u_{i\nu}$ $(i = 1, \cdots, n; \nu = 0, \cdots, m)$ be arbitrary real constants. Consider regular single-valued harmonic functions $u(z)$ on W with

$$\partial^\nu u(\zeta_i)/\partial x^\nu = u_{i\nu}. \tag{6}$$

Here $z = x + iy$ is the local uniformizer at ζ_i. We are interested in the interpolation problem of constructing a harmonic function satisfying (6) and with a smallest possible value of the Dirichlet integral. If $m = 0$, we are dealing with the simplest case where the $u(\zeta_i)$ only are given. If $n = 1$, we have functions with values and derivatives, up to the mth order, prescribed at one point only.

We shall show (no. 24) that the principal functions p_0, p_1 again provide us with the solution:

The function

$$u_0 = p_0 - p_1, \tag{7}$$

for properly chosen b_ν, c, satisfies the conditions (6) *and minimizes $D(u)$. The value of the minimum is $\mu_0 - \mu_1$, and the deviation from this minimum is given* *by the Dirichlet integral of* $u-u_0$:

$$D(u) = \mu_0 - \mu_1 + D(u - u_0). \tag{8}$$

The possibility of a proper choice of the b_ν, c will be fully characterized by the nonvanishing of a determinant (no. 23).

If W is planar, we shall also consider single-valued analytic functions $V(z)$ with prescribed values of $V^{(\nu)}(\zeta_i)$, $\nu=1, \cdots, m$. Then the analytic completions of suitably modified functions p_0, p_1 will furnish the solution, which exists if and only if the boundary of W is not AD-removable (no. 28).

With regard to the vast literature concerning interpolation problems in the unit circle, we refer here to the well-known results of Walsh [21], Pick [15], Nevanlinna [13; 14] and Weyl [22]. These authors discussed the interpolation problem from the viewpoint of minimizing mean values. The question of minimizing the Dirichlet integral was first treated by Kubota [6; 7], Takenaka [20] and, for more general plane regions, by Lokki [9–12], Garabedian and Schiffer [4], and Lehto [8]. The prescribed quantities in these investigations were the values of the functions ($m=0$), while Bergman [3] also considered values of derivatives at a fixed point ($n=1$). Concerning extremal problems for differentials, the reader is referred to the comprehensive study [1] by Ahlfors.

In what follows, any finite values of n, m will be permitted and the region of existence is allowed to be an arbitrary open Riemann surface. Our approach, sketched above, is based on the extremal method introduced in [17].

§1. Reduction theorem

4. Let W be an arbitrary open Riemann surface and C a class of functions on W. Consider an exhaustion $\{W_n\}$ of W by compact subregions W_n, each bounded by a finite set β_n of analytic Jordan curves. On W_n, a class C_n of functions is given with the property that the restriction to W_{n-1} of $p\in C_n$ belongs to C_{n-1}. It is understood that this holds, correspondingly, for W, W_n and C, C_n.

For W_n and $p\in C_n$, a functional $m(W_n, p)$ is given, such that, for $p'\in C_n$ tending uniformly on W_n to $p''\in C_n$, we have $m(W_n, p')\to m(W_n, p'')$. The functional $m(W, p)$ for W and $p\in C$ is defined by

$$m(W, p) = \lim_{n\to\infty} m(W_n, p), \tag{1}$$

the existence of the limit being postulated.

The problem is to find a function p_W which minimizes $m(W, p)$ in C, and to determine properties of p_W and $m(W, p_W)$.

5. We shall establish the following

Reduction theorem. *Suppose that the functionals* $m(W_n, p)$ *defined above satisfy the following conditions*:

1°. *The problem has a solution for* W_n, *that is, there exists a function* p_n *in* C_n *with*

(2) $$\min_{p \in C_n} m(W_n, p) = m(W_n, p_n).$$

2°. *The functional m is an increasing function of the region:*

(3) $$m(W_n, p) \leqq m(W_{n+1}, p)$$

for $p \in C_{n+1}$.

3°. *The family* $\{p_n\}$ *is normal, the limiting functions on W belonging to C.*

Then every limiting function, $p_W = \lim_{n\to\infty} p_n$, *say, is a solution of the extremal problem for W,*

(4) $$\min_{p \in C} m(W, p) = m(W, p_W),$$

and the value of the minimum is

(5) $$m(W, p_W) = \lim_{n\to\infty} m(W_n, p_n).$$

Proof. We first observe that, by definition,

$$m(W, p_W) = \lim_{n\to\infty} m(W_n, p_W) = \lim_{n\to\infty} \lim_{k\to\infty} m(W_n, p_k).$$

Consequently, by virtue of condition 2°,

(6) $$m(W, p_W) \leqq \lim_{n\to\infty} m(W_n, p_n).$$

On the other hand, we have for $p \in C$ the inequalities

$$m(W_n, p_n) \leqq m(W_n, p) \leqq m(W, p).$$

As this holds for each W_n and $p \in C$, we deduce that

(7) $$\lim_{n\to\infty} m(W_n, p_n) \leqq \inf m(W, p) \leqq m(W, p_W).$$

The statements (4) and (5) now follow on combining (6) and (7).

§2. A THEOREM ON MINIMA OF FUNCTIONALS

6. We consider the class of single-valued harmonic functions p with a finite number of given singularities on an open Riemann surface W. It is no restriction to assume that the singularities ζ_i $(i=1, \cdots, n)$ are centers of disjoint parametric disks K_i. The functions p have the following expansions in K_i:

(1) $$p = \operatorname{Re}\left\{\sum_0^\infty a_\nu z^\nu + \lambda\cdot\left(\sum_1^m b_\nu z^{-\nu} - c \log |z|\right)\right\}.$$

Here a_ν, b_ν are complex, c real coefficients, and λ a real parameter. The class of functions (1) with fixed b_ν, c and subsequently fixed λ shall be denoted by C_λ. In particular, C_0 is the class of all single-valued regular harmonic functions p on W.

For simplicity, we shall not use additional indices i to distinguish the a_ν, b_ν, c, m and z corresponding to the various K_i. To avoid unnecessary constants, we also shall assume that the functions p have been normalized, by adding suitable real constants, so as to have Re $\{a_0\}=0$ at ζ_1. Moreover, the coefficients c, whose reality is implied by the single-valuedness of p, are supposed to satisfy the condition $\sum_{i=1}^{n} c=0$. We shall use the abbreviation

$$\mu = 2\pi \sum_{i=1}^{n} \operatorname{Re}\left[ca_0 + \sum_{\nu=1}^{m} \nu b_\nu a_\nu\right]. \tag{2}$$

7. Set $\lambda = h+k$ with real h, k, and consider the following functional of p:

$$m(W, p) = \int_\beta p\,dp^* + (k-h)\mu. \tag{3}$$

Here the integral along the ideal boundary β of W is defined as the limit of integrals taken along boundaries of exhausting regions.

The following theorem will be of basic importance in our reasoning:

THEOREM. *There exists a uniquely determined function p_{hk} in C_λ which minimizes $m(W, p)$,*

$$\min_{p\in C_\lambda} m(W, p) = m(W, p_{hk}). \tag{4}$$

For various h, k, the functions p_{hk} are interrelated by

$$p_{hk} = hp_0 + kp_1, \tag{5}$$

where p_0, p_1 stand for $p_{1,0}$, $p_{0,1}$, respectively.

The minimum value of $m(W, p)$ can be expressed solely in terms of the a_ν-coefficients of p_0, p_1:

$$m(W, p_{hk}) = k^2\mu_1 - h^2\mu_0, \tag{6}$$

μ_0, μ_1 being the values of μ for p_0, p_1. The deviation from this minimum is given by the Dirichlet integral:

$$m(W, p) = k^2\mu_1 - h^2\mu_0 + D(p - p_{hk}). \tag{7}$$

The proof, based on the reduction theorem, will be given in nos. 8–17.

8. We first have to show that condition 1° of the reduction theorem is satisfied, that is, the problem can be solved for a Riemann surface W whose boundary β consists of a finite number of analytic Jordan curves.

In this case there exist in C_1 two uniquely determined functions p_0, p_1, defined by the following properties:

$$\begin{cases} \partial p_0/\partial n = 0 \text{ on } \beta, \\ \quad p_1 = \text{const. on } \beta. \end{cases} \tag{8}$$

These functions can be easily constructed by the linear operator method [16], for the basic condition $\int ds^* = 0$ is obviously satisfied for both p_0 and p_1. This implies that the functions in the sequel can be formed purely constructively.

We set

$$p_{hk} = hp_0 + kp_1, \tag{9}$$

and shall show that the deviation formula (7) holds. To begin with, we have

$$D(p - p_{hk}) = \int_\beta pdp^* + \int_\beta p_{hk}dp_{hk}^* - \int_\beta pdp_{hk}^* - \int_\beta p_{hk}dp^*. \tag{10}$$

9. The second integral on the right simplifies immediately:

$$\int_\beta p_{hk}dp_{hk}^* = hk\int_\beta p_0dp_1^* = hk\int_\beta p_0dp_1^* - p_1dp_0^*.$$

By Green's formula, the last integral can be transferred to the peripheries α_i of the K_i:

$$\int_\beta p_{hk}dp_{hk}^* = hk\sum_i \int_{\alpha_i} p_0dp_1^* - p_1dp_0^* . \tag{11}$$

If we set $p_q = u_q + s_q$ $(q=0, 1)$ in K_i, with

$$u_q = \operatorname{Re}\left\{\sum_0^\infty a_{q\nu}z^\nu + \sum_1^m b_\nu z^{-\nu}\right\},$$

$$s_q = c\log(1/|z|),$$

then

$$\int_{\alpha_i} p_0dp_1^* - p_1dp_0^* = \int_{\alpha_i} u_0du_1^* - u_1du_0^* + \int_{\alpha_i}(u_0 - u_1)ds_q^*. \tag{12}$$

As $U_0 = u_0 + iu_0^*$ and $U_1 = u_1 + iu_1^*$ are single-valued in K_i, one obtains, on integrating by parts,

$$\int_{\alpha_i} u_0du_1^* - u_1du_0^* = \int_{\alpha_i} u_0du_1^* + u_0du_1^* = \operatorname{Im}\int_{\alpha_i} U_0dU_1. \tag{13}$$

The integral here can be written

$$\int_{\alpha_i} U_0 dU_1 = \int_{\alpha_i} \left(\sum_0 a_{0\nu} z^\nu + \sum_1 b_\nu z^{-\nu} \right) \left(\sum_1 \nu a_{1\nu} z^{\nu-1} - \sum_1 \nu b_\nu z^{-\nu-1} \right) dz$$

$$(14) \qquad = \int_{\alpha_i} \left(- \sum_1 \nu b_\nu a_{0\nu} + \sum_1 \nu b_\nu a_{1\nu} \right) \frac{dz}{z}$$

$$= 2\pi i \sum_1 \nu b_\nu (a_{1\nu} - a_{0\nu}).$$

The latter part of (12) has the value

$$(15) \quad \int_{\alpha_i} (u_0 - u_1) ds_q^* = -c \int_{\alpha_i} (u_0 - u_1) d \arg z = 2\pi \operatorname{Re} \{ c(a_{10} - a_{00}) \},$$

and we find, on combining (11)–(15), that

$$(16) \quad \int_\beta p_{hk} dp_{hk}^* = 2\pi hk \sum_i \operatorname{Re} \left\{ c(a_{10} - a_{00}) + \sum_1 \nu b_\nu (a_{1\nu} - a_{0\nu}) \right\}$$

$$= hk(\mu_1 - \mu_0).$$

10. For the third integral in (10), we have similarly

$$\int_\beta p dp_{hk}^* = k \int_\beta p dp_1^* - p_1 dp^* = k \sum_i \int_{\alpha_i} p dp_1^* - p_1 dp^*.$$

On setting $p = u + s$ with

$$u = \operatorname{Re} \left\{ \sum_0^\infty a_\nu z^\nu + (h + k) \sum_1^m b_\nu z^{-\nu} \right\},$$

$$s = (h + k) c \log (1/|z|),$$

one obtains

$$\int_{\alpha_i} p dp_1^* - p_1 dp^* = \int_{\alpha_i} u du_1^* - u_1 du^* + \int_{\alpha_i} u ds_1^* - u_1 ds^* .$$

Here

$$\int_{\alpha_i} u du_1^* - u_1 du^* = \operatorname{Im} \int_{\alpha_i} U dU_1$$

with

$$\int_{\alpha_i} U dU_1 = \int_{\alpha_i} \left[\sum_0 a_\nu z^\nu + (h + k) \sum_1 b_\nu z^{-\nu} \right] \left[\sum_1 \nu a_{1\nu} z^{\nu-1} - \sum_1 \nu b_\nu z^{-\nu-1} \right] dz$$

$$= 2\pi i \sum_1 \nu b_\nu [(h + k) a_{1\nu} - a_\nu],$$

and

$$\int_{\alpha_i} uds_1^* - u_1 ds^* = -2\pi \operatorname{Re}\{ca_0 - (h+k)ca_{10}\}.$$

Consequently,

$$\begin{aligned}(17)\quad \int_\beta pdp_{hk}^* &= 2\pi k \sum_i \operatorname{Re}\left\{c[(h+k)a_{10} - a_0] + \sum_1 \nu b_\nu[(h+k)a_{1\nu} - a_\nu]\right\} \\ &= k[(h+k)\mu_1 - \mu].\end{aligned}$$

11. Finally, for the last term in (10), we have

$$\begin{aligned}\int_\beta p_{hk} dp^* &= h\int_\beta p_0 dp^* - pdp_0^* = h\sum_i \int_{\alpha_i} p_0 dp^* - pdp_0^* \\ &= h\sum_i\left(\int_{\alpha_i} u_0 du^* - udu_0^* + \int_{\alpha_i} u_0 ds^* - uds_0^*\right).\end{aligned}$$

This gives, as in no. 10,

$$\begin{aligned}(18)\quad \int_\beta p_{hk} dp^* &= 2\pi h \sum_i \operatorname{Re}\left\{c[a_0 - (h+k)a_{00}] + \sum_1 \nu b_\nu[a_\nu - (h+k)a_{0\nu}]\right\} \\ &= h[\mu - (h+k)\mu_0].\end{aligned}$$

12. On summing up, we find from (10), (16), (17), (18) that

$$(19)\qquad \int_\beta pdp^* + (k-h)\mu = (k^2\mu_1 - h^2\mu_0) + D(p - p_{hk}).$$

Since $D(p-p_{hk})$ is nonnegative and vanishes for $p=p_{hk}$, we conclude that the functional on the left is minimized by p_{hk}, and the value of the minimum is $k^2\mu_1 - h^2\mu_0$. Thus, condition 1° of the reduction theorem is fulfilled and the deviation formula (7) holds in the special case where β consists of a finite number of analytic Jordan curves.

13. Condition 2° of the reduction theorem is easily verified. In fact, for every $p \in C_{n+1}$,

$$(20)\qquad m(W_{n+1}, p) - m(W_n, p) = \int_{\beta_{n+1}-\beta_n} pdp^*,$$

where the integral equals the Dirichlet integral of p over $W_{n+1} - W_n$, thus being nonnegative.

14. To prove that condition 3° of the reduction theorem is satisfied, let now W be an arbitrary open Riemann surface, with an exhaustion $\{W_n\}$ by compact subregions W_n, bounded by finite sets β_n of analytic Jordan curves. Let p^n be the function $p_{hk} = hp_0 + kp_1$ for W_n. More precisely, p^n minimizes the functional

$$m(W_n, p) = \int_{\beta_n} p\,dp^* + (k-h)\mu$$

in the class C_n of functions p harmonic on W_n except at the singularities ζ_i. Correspondingly, we write $p_0^n, p_1^n, \mu_0^n, \mu_1^n$ for p_0, p_1, μ_0, μ_1 on W_n. If D_N denotes the Dirichlet integral over a fixed W_N, then, by (19) for W_N, and for $n>N$,

$$\begin{aligned} D_N(p^n - p^N) &= m(W_N, p^n) - m(W_N, p^N) \\ &\leqq m(W_n, p^n) - m(W_N, p^N) \\ &= k^2(\mu_1^n - \mu_1^N) - h^2(\mu_0^n - \mu_0^N). \end{aligned} \tag{21}$$

On setting $h=1$, $k=0$ and then $h=0$, $k=1$ in (19) for W_N, we infer that $\mu_1^N \leqq \mu \leqq \mu_0^N$ for functions p with $\int_{\beta_N} p\,dp^* \leqq 0$. The latter condition is fulfilled, in particular, by p_0^n, p_1^n, for $\int_{\beta_N} p_0^n dp_0^{n*} \leqq \int_{\beta_n} p_0^n dp_0^{n*} = 0$, and the same is true of p_1^n. Consequently,

$$D_N(p^n - p^N) \leqq (h^2 + k^2)(\mu_0^N - \mu_1^N).$$

Here the right-hand side is independent of n, and we conclude that the family p^n is normal. Thus condition 3° of the reduction theorem is fulfilled.

15. The reduction theorem now provides us with the existence on W of a limiting function p_{hk} of a subsequence, say again $\{p^n\}$, with the property

$$\min_{p \in C_\lambda} m(W, p) = m(W, p_{hk}). \tag{22}$$

Moreover, the value of this minimum is

$$m(W, p_{hk}) = k^2\mu_1 - h^2\mu_0,$$

where μ_1, μ_0 are the limits of μ_1^n, μ_0^n.

16. The deviation formula will now be based on the minimum property of p_{hk}. If we set $p - p_{hk} = u$ for $p \in C_\lambda$, then, for real ϵ,

$$\begin{aligned} m(W_n, p_{hk} + \epsilon u) &= m(W_n, p_{hk}) + \epsilon^2 D_n(u) \\ &\quad + \epsilon\left[\int_{\beta_n} (p_{hk}du^* + u\,dp_{hk}^*) + (k-h)(\mu - \mu_{hk})\right]. \end{aligned}$$

Suppose first that $D(u) < \infty$ over W. Then the first three terms of this formula tend to finite limits for $n \to \infty$, and so does, a fortiori, the expression I_n, say, in the brackets; set $I = \lim I_n$. It follows that

$$m(W, p)_{hk} + \epsilon u = m(W, p_{hk}) + \epsilon^2 D(u) + \epsilon I.$$

By the minimum property, $dm/d\epsilon = 0$ for $\epsilon = 0$, whence $I = 0$. On setting $\epsilon = 1$ we have the desired deviation formula

$$m(W, p_{hk} + \epsilon u) = m(W, p_{hk}) + D(p - p_{hk}).$$

This holds, trivially, in the case $D(p-p_{hk})=\infty$ as well. In fact, if W_0 is a compact subregion containing in its interior the singularities ζ_i of p, then the Dirichlet integrals D^0 extended over $W-W_0$ satisfy the triangle inequality

$$(D^0(p - p_{hk}))^{1/2} \leqq (D^0(p))^{1/2} + (D^0(p_{hk}))^{1/2}.$$

Since, by (16) for W, $\int_\beta p_{hk} dp_{hk}^*$ is always finite, so is $D^0(p_{hk})$, while the finiteness of $m(W, p)$ would imply that of $\int_\beta p dp^*$ and of $D^0(p)$. But then $D(p-p_{hk})$ would be finite, contrary to the assumption. We infer that $m(W, p)$ must be infinite, and the deviation formula in this degenerate form continues to hold.

17. It remains to show that the minimizing function p_{hk} is unique. Suppose p', p'' are two minimizing functions. Then

$$D(p' - p'') = m(W, p') - m(W, p'') = 0$$

and $p'-p''$ is constant. By the normalization Re $\{a_0\}=0$ at the center ζ_1 of K_1, this constant vanishes, and we have $p'\equiv p''$. The proof of our theorem (no. 7) is herewith complete.

§3. Interpolation of harmonic functions

18. We now turn to the following

Interpolation problem. *Given an open Riemann surface W, and a finite number of points ζ_i $(i=1, \cdots, n)$ on W. Minimize the Dirichlet integral $D(u)$ among all harmonic functions $u(z)$, $z=x+iy$, on W with prescribed values, $u_{i\nu}$, of $\partial^\nu u(\zeta_i)/\partial x^\nu$, $\nu=0, \cdots, m$.*

Again we may choose the points ζ_i to be the centers of disjoint parametric disks K_i. Then the $u(z)$ have the expansions

$$u(z) = \text{Re}\left\{\sum_{\nu=0}^{\infty} \overset{i}{a}_\nu \overset{i}{z}{}^\nu\right\}, \tag{1}$$

the indices i now being written down to indicate the individual parametric disks. In terms of these expansions, the prescribed quantities are

$$\text{Re}\,\{\overset{i}{a}_\nu\} = u_{i\nu}$$

19. The theorem of no. 7 has the following immediate consequence:

The function $u_0=p_0-p_1$ gives to the Dirichlet integral the minimum

$$\min D(u) = \mu_0 - \mu_1 \tag{2}$$

among all harmonic functions u on W with

$$\mu = \mu_0 - \mu_1. \tag{3}$$

The deviation from this minimum is measured by

$$D(u - u_0) = D(u) - D(u_0). \tag{4}$$

In fact, the class determined by (3) is certainly not empty, for u_0 belongs to it. The functions $p \in C_0$ ($\lambda=0$) are regular harmonic, and in the case $h=1$, $k=-1$, formula (7), no. 7, can be written

$$D(u) - 2\mu = \mu_1 - \mu_0 + D(u - u_0).$$

For $\mu=\mu_0-\mu_1$, the statement follows.

20. Condition (3) is fulfilled, in particular, by the functions u with

$$\text{Re}\,\{a_\nu^i\} = \text{Re}\,\{a_{0\nu}^i - a_{1\nu}^i\}, \tag{5}$$

$\nu=0, \cdots, m$, if the b_ν are taken real. Thus we have the solution of our interpolation problem for the special values on the right. If these values can be arbitrarily prescribed, by properly choosing the b_ν^i and c^i in the expansions

$$p = \text{Re}\left\{\sum_0^\infty a_\nu^i z^\nu + \lambda\cdot\left(\sum_1^m b_\nu^i z^{-\nu} - c^i \log|z|\right)\right\}, \tag{6}$$

then the solution of our problem is complete. Our task amounts to studying when such a choice is possible.

21. The functions p_0, p_1 can be expressed as linear combinations of certain "elementary functions". To see this, we choose the $b_\nu^i=0$, $c^1=-1$, $c^j=1$ (j fixed $\neq 1$), the other $c^i=0$. Then we have a class $\{s^j\}$ of harmonic functions with two logarithmic poles only; let s_0^j, s_1^j be the functions p_0, p_1 in this special class. Similarly, take the $c^i=0$, $b_k^j=1$ (j, k fixed), the other $b_\nu^i=0$. Now we have a single pole at a fixed ζ_j, of order $k(\leqq m)$, and we denote the corresponding functions p_0, p_1 by t_{k0}^j, t_{k1}^j.

In the general class C_λ with arbitrary real b_ν^i, c^i the functions p_0, p_1 can be written

$$\begin{aligned} p_0 &= \sum_{j=2}^n c^j s_0^j + \sum_{j=1}^n \sum_{k=1}^m b_k^j t_{k0}^j, \\ p_1 &= \sum_{j=2}^n c^j s_1^j + \sum_{j=1}^n \sum_{k=1}^m b_k^j t_{k1}^j. \end{aligned} \tag{7}$$

Indeed, for an approximating region W_n, the functions p_0, p_1 are uniquely determined in C_1 by the boundary conditions $\partial p_0/\partial n=0$, $p_1=$const. on β_n. As these conditions are fulfilled by the s_0^j, s_1^j and t_{k0}^j, t_{k1}^j, so they are by the linear combinations (7) which therefore constitute p_0, p_1. For $W_n \to W$, all these functions converge to unique limiting functions, and the relations (7) remain valid on W.

22. The difference

$$p_0 - p_1 = \sum_{j=2}^n c^j(s_0^j - s_1^j) + \sum_{j=1}^n \sum_{k=1}^m b_k^j(t_{k0}^j - t_{k1}^j) \tag{8}$$

has the expansion

$$p_0 - p_1 = \text{Re}\left\{\sum_{\nu=0}^{\infty}(\overset{i}{a}_{0\nu} - \overset{i}{a}_{1\nu})z^{\nu}\right\} \tag{9}$$

in K_i. For the s-functions we write in K_i

$$\overset{j}{s}_0 - \overset{j}{s}_1 = \text{Re}\left\{\sum_{\nu=0}^{\infty}\overset{ji}{\sigma}_{\nu} z^{\nu}\right\} \tag{10}$$

with $j=2, \cdots, n; i=1, \cdots, n$. For the t-functions we set, correspondingly,

$$\overset{j}{t}_{k0} - \overset{j}{t}_{k1} = \text{Re}\left\{\sum_{\nu=0}^{\infty}\overset{ji}{\tau}_{k\nu} z^{\nu}\right\}. \tag{11}$$

Then, by (8),

$$\text{Re}\left\{\sum_{j=2}^{n}\overset{j}{c}\,\overset{ji}{\sigma}_{\nu} + \sum_{j=1}^{n}\sum_{k=1}^{m}\overset{j}{b}_k\overset{ji}{\tau}_{k\nu}\right\} = \text{Re}\left\{\overset{i}{a}_{0\nu} - \overset{i}{a}_{1\nu}\right\} \tag{12}$$

for $i=1, \cdots, n; \nu=0, \cdots, m$.

23. In the relations (12), the σ's and τ's are known constants, and the unknown real constants c's and b's are to be so determined that the right-hand sides assume the prescribed values $u_{i\nu}$. By the normalization $R\{a_0^1\}=0$ for all p, s, t, no equation corresponds to $i=1$, $\nu=0$. Thus we are dealing with $n-1+nm$ equations with the same number of unknowns. A solution exists if and only if the determinant

$$\Delta = \left|\ \text{Re}\left\{\overset{ji}{\sigma}_{\nu}\right\}, \text{Re}\left\{\overset{ji}{\tau}_{k\nu}\right\}\right| \tag{13}$$

is different from zero. Here the rows are formed for varying j, k, the columns for i, ν.

24. We have thus arrived at the solution of our interpolation problem:

THEOREM. *Given an arbitrary Riemann surface W, and points ζ_i ($i=1, \cdots, n$) on W, let $u_{i\nu}$ ($\nu=0, \cdots, m$) be arbitrary finite real constants. Let $\{u\}$ be the class of regular single-valued harmonic functions on W with*

$$\partial^{\nu}u(\zeta_i)/\partial x^{\nu} = u_{i\nu}. \tag{14}$$

There exists, in $\{u\}$, a function u_0 which minimizes the Dirichlet integral,

$$\min D(u) = D(u_0), \tag{15}$$

if and only if, in (13), $\Delta\neq 0$.

The minimizing function is unique and has the expression

$$u_0 = p_0 - p_1, \tag{16}$$

where p_0, p_1 are taken in that class C_1 whose coefficients b^i_{ν}, c^i give the prescribed values $u_{i\nu}$ to the expressions (12).

In terms of these coefficients, the minimum of $D(u)$ is $\mu_0-\mu_1$, and the devia-

tion from this minimum is measured by the Dirichlet integral of $u-u_0$:

$$D(u) = \mu_0 - \mu_1 + D(u - u_0). \tag{17}$$

The remark might be added that the above reasoning remains valid verbatim in the case where the values of $u(\zeta_i)$ and $U^{(\nu)}(\zeta_i)$, $\nu=1, \cdots, m$, are prescribed, with $u=\text{Re}\,\{U\}$.

§4. Interpolation of analytic functions

25. In what follows, W signifies a planar Riemann surface. We consider single-valued analytic functions $Q=q+iq^*$ on W, with expansions

$$Q = \sum_0^\infty \overset{i}{a}_\nu \overset{\nu}{z} + \lambda \cdot \sum_1^m \overset{i}{b}_\nu \overset{i}{z}^{-\nu} \tag{1}$$

about the ζ_i. For fixed complex b_ν^i and subsequently fixed real λ, the class of functions Q with normalization $a_0^1=0$ shall be designated by G_λ. The symbol μ now stands for

$$\mu = 2\pi \,\text{Re} \sum_{i=1}^n \sum_{\nu=1}^m \nu \overset{i}{b}_\nu \overset{i}{a}_\nu. \tag{2}$$

If W is compact and its boundary β consists of a finite number of analytic Jordan curves γ, there are unique functions Q_0, Q_1 in G_1, determined by the conditions

$$\begin{aligned} \partial q_0/\partial n &= 0 \quad \text{on } \beta, \\ q_1 &= r_\gamma \quad \text{on } \gamma. \end{aligned} \tag{3}$$

The constants r_γ are so chosen that

$$\int_\gamma dq_1^* = 0 \tag{4}$$

along each contour $\gamma \subset \beta$.

For an open W, q_0, q_1 appear as limiting functions of the q_0, q_1 for exhausting subregions W_n. The reasoning in §§2, 3 remains valid throughout when p, p_0, p_1 are replaced by q, q_0, q_1, and the basic formula now takes the form

$$\int_\beta q\,dq^* + (k-h)\mu = k^2\mu_1 - h^2\mu_0 + D(q - q_{hk}). \tag{5}$$

26. For regular single-valued analytic functions V on W with expansions

$$V = \sum_{\nu=0}^\infty \overset{i}{a}_\nu \overset{\nu}{z} \tag{6}$$

in K_i and with $\mu=\mu_0-\mu_1$, we conclude from (5):

The function $V_0=Q_0-Q_1$ *has the minimum property*

$$D(V) = \mu_0 - \mu_1 + D(V - V_0). \tag{7}$$

The condition $\mu=\mu_0-\mu_1$ is satisfied, in particular, by those functions for which

$$\overset{i}{a}_\nu = \overset{i}{a}_{0\nu} - \overset{i}{a}_{1\nu}, \tag{8}$$

$\nu=1, \cdots, m$. If here the right sides can be prescribed at will, by properly choosing the complex b_ν^i, then we are in possession of a function V with arbitrarily given values of the $V^{(\nu)}(\zeta_i)$, and with a minimal $D(V)$.

27. In analogy to no. 21, let Q_{k0}^j, Q_{k1}^j be the functions Q_0, Q_1 with the sole singularity $1/z^k$ at ζ_j. For the corresponding functions $Q_{k0}^{j\phi}$, $Q_{k1}^{j\phi}$ with poles $e^{i\phi}/z^k$ at ζ_j, we have

$$\begin{aligned} Q_{k0}^{j\phi} &= Q_{k0}^{j} \cos\phi + iQ_{k1}^{j} \sin\phi, \\ Q_{k1}^{j\phi} &= iQ_{k0}^{j} \sin\phi + Q_{k1}^{j} \cos\phi. \end{aligned} \tag{9}$$

In fact, on a compact W bounded by analytic Jordan curves, the combinations on the right have the same singularity as $Q_{k0}^{j\phi}$, $Q_{k1}^{j\phi}$. Moreover, the real part of the first combination has vanishing normal derivative on the boundary, while that of the second is constant on each contour. For an open W, the relations (9) then follow by an exhaustion $W_n \to W$.

We obtain from (9),

$$Q_{k0}^{j\phi} - Q_{k1}^{j\phi} = e^{-i\phi}(Q_{k0}^{j} - Q_{k1}^{j}), \tag{10}$$

and conclude that (8), no. 22, is now replaced by

$$Q_0 - Q_1 = \sum_{j=1}^{n}\sum_{k=1}^{m} \overline{b_k^j}(Q_{k0}^{j} - Q_{k1}^{j}), \tag{11}$$

the bar indicating the complex conjugate. Equation (12), no. 22, takes the form

$$\sum_{j=1}^{n}\sum_{k=1}^{m} \overline{b_k^j}\,\tau_{k\nu}^{ji} = \overset{i}{a}_{0\nu} - \overset{i}{a}_{1\nu}. \tag{12}$$

For prescribed right sides, we now have nm equations for the same number of unknowns b_j^k.

28. We proceed to demonstrate that, in the present case, the solvability of the system is characterized by the existence of functions AD on W, that is, single-valued nonconstant analytic functions with a finite-Dirichlet integral. The class of Riemann surfaces without functions AD is said to have an AD-removable boundary and is denoted by O_{AD}. The necessity of the condition $W \notin O_{AD}$ is obvious, for we must have functions on W whose Dirichlet integral could be minimized.

To establish the sufficiency, suppose that there is indeed a function AD on W. Then W can be mapped onto a plane region which contains the point at infinity and whose complement has a positive area (Ahlfors-Beurling [2]). Let $w(z)=u+iv$ be the mapping function, and denote the images of W, β, ζ_i by W, β, w_i, the complement of W, by H. By (7), we have on setting $h=k=1$ in (5),

$$D(Q_0 - Q_1) = \mu_0 - \mu_1 = - \int_\beta (q_0 + q_1) d(q_0 + q_1)^* = \max_{G_2} \int_{-\beta} q dq^*.$$

The function

$$Q(w) = 2 \sum_{i=1}^{n} \sum_{\nu=1}^{m} \bar{b}_k^j (w - w_i)^{-\nu}$$

belongs to G_2 if the $\bar{b}_k^j$ are so chosen that $Q(w(z))$ has the given coefficients $2b_k^j$ at the ζ_i. Since $Q(w)$ is nonconstant, its Dirichlet integral over H does not vanish,

$$D_H(Q) = \int_{-\beta} q dq^* > 0,$$

and it follows that $D(Q_0 - Q_1) > 0$. From this we infer that $Q_0 - Q_1$ and, a fortiori, the linear combination on the right in (11) is not constant. Consequently, the functions $Q_{k0}^j - Q_{k1}^j$ are linearly independent; the system (12) has a solution for any prescribed values on the right.

29. We have established the following result:

THEOREM. *On a planar Riemann surface W, let $\{V\}$ be the class of regular single-valued analytic functions with arbitrarily prescribed values*

(13) $$d^{(\nu)} V(\zeta_i)/dz^\nu = V_{i\nu}$$

at given points ζ_i, $i=1, \cdots, n$; $\nu=1, \cdots, m$. There exists a function V_0 in $\{V\}$ with the property

(14) $$\min_{\{V\}} D(V) = D(V_0)$$

if and only if $W \notin O_{AD}$.

The minimizing function is

(15) $$V_0 = Q_0 - Q_1$$

where Q_0, Q_1 are taken in the class G_1 determined by the coefficients b_ν^i which give the values $V_{i\nu}$ to the quantities (12). *The value of the minimum and the deviation from the minimum are given, in terms of these coefficients, by the formula*

(16) $$D(V) = \mu_0 - \mu_1 + D(V - V_0).$$

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